JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 29, No. 5, September–October 2006

Game Optimal Support Time of a Medium Range Air-to-Air Missile

Janne Karelahti,* Kai Virtanen,† and Tuomas Raivio‡
Helsinki University of Technology, FIN-02015 HUT, Espoo, Finland

DOI: 10.2514/1.17910

This paper formulates a support time game arising in one-on-one air combat with medium range air-to-air missiles. The game model provides game optimal support times of the missiles that can receive target information from the launching aircraft for selectable support times. The payoffs of the game are formulated as a weighted sum of the probabilities of hit to the adversary and own survival. Under suitable simplifying assumptions, a Nash equilibrium of the game can be computed by an iterative search involving a series of optimal control problems. For practical situations, an approximate real time computation scheme is introduced. The constructed model and the scheme are illustrated by numerical examples.

I. Introduction

In this paper, we consider a game setting between two hostile aircraft, each equipped with one medium range air-to-air missile. The guidance of such a missile typically consists of three phases. At first, the launching aircraft relays target information to the missile in the support phase. In the second phase, the aircraft evades, and the missile continues in a silent extrapolation mode toward the expected rendezvous point. Finally, in the endgame, the missile switches on its own radar and tries to lock its seeker on the target.

The pilots can select the lengths of their support phases freely. Prolonging the support phase shortens the extrapolation phase, which increases the probability that the missile's seeker will lock on the target that in turn increases the probability of hit. Nevertheless, the probability of survival decreases because supporting requires flying toward a missile potentially delivered by the adversary. The above-mentioned probabilities depend on the actions of the adversary as well. Hence, the game problem is to select the maneuvers and support times that maximize these probabilities under the assumption that the adversary behaves rationally.

Despite its central position in modern air combat, this specific problem seems to have received only a limited attention in the open literature. In general, missile duels have been modeled using concepts of differential game theory [1], artificial intelligence, and simulation [2]. Differential game formulations provide game optimal controls of the players against the optimal controls of the other player. In [3], the computation of the largest possible firing range for an optimally guided missile in pursuit-evasion game framework [4] is addressed. In [5], a two-target differential game [6] based model with all-aspect guided missiles and simple vehicle models is analyzed. In [7], a zero-sum differential game formulation of a three-dimensional duel with fire-and-forget type missiles is presented. In [8], zero- and nonzero-sum differential game formulations of a three-dimensional duel with a requirement that the missile must be supported until its range to the target is equal to a predefined lock-on range are introduced.

Artificial intelligence techniques combined with pursuit-evasion game solutions have been utilized as part of pilot decision support systems the purpose of which is to recommend favorable launch and evasion moments [9,10], that is, the support time. In these systems, the set of available strategies is predefined and the adversary behaves according to a specified feedback rule. The strategy set is always limited and the feedback rule of the adversary is likely suboptimal, but all the computation can be done in real time.

In [11], a missile duel is solved by simulation. Instead of selecting controls of the aircraft, the players apply suboptimal feedback guidance laws. The best guidance laws of the players for a particular initial state are obtained as a solution of a bimatrix game. Again, the missiles are considered as a fire-and-forget type. In batch air combat simulators [12], the duel is simulated several times with randomly generated values of the variables describing uncertainties of the combat. This results in a distribution of the number of lost aircraft for each side. Simulators provide a means for analyzing the efficiency of given support and evasion maneuvers with a fixed support time.

As far as the authors know, the combat model presented in the paper is the first game formulation in which the support times of the players are taken into account explicitly. This allows the determination of game optimal support times without the use of predetermined heuristic rules. For computational analysis, the conflicting objectives of the pilot, that is, the probabilities of hit and survival, are here transformed into a scalar-valued payoff function of the game by forming a weighted sum of these probabilities. The weights represent the risk attitude of the pilot.

The probabilities themselves possess a difficult modeling task, and some simplifications are needed. A successful hit and its complement, survival, depend on two probabilities: the probability that the missile will lock on the target and the probability that the hit is effective. In this paper, we assume that the former probability depends on the angular error accumulating during the target extrapolation in the silent phase. For the latter probability, the closing velocity of the missile is used as the explaining factor. Many other aspects affect these probabilities, too, but these are considered as the most significant ones.

As such, the game is a nonzero-sum differential game. The trajectories and payoffs after the support phase are, however, decoupled, as both players minimize independently the closing velocity of the missile to minimize the probability of an effective hit. To completely decouple the dynamics, we make an assumption, justified by air combat experience [13], that during the support phase both the aircraft fly at the gimbal limit of their radars. Then, the controls during the support phase depend only on the initial state of the game and can be fixed beforehand. Thus, in addition to optimal missile evasion maneuver, only the support time needs to be decided, and the payoff functions of the game can be evaluated based on the solutions of a set of closing-velocity minimizing optimal control

Received 27 May 2005; revision received 2 November 2005; accepted for publication 3 January 2006.

*Researcher, Systems Analysis Laboratory, P.O. Box 1100; janne.karelahti@hut.fi.

†D.Sc., Researcher, Systems Analysis Laboratory, P.O. Box 1100; kai.virtanen@hut.fi.

‡D.Sc., Docent, Systems Analysis Laboratory, P.O. Box 1100; tuomas.raivio@hut.fi.

problems. After evaluating the payoffs, a Nash equilibrium of the support time game can be solved with, for example, a best response algorithm.

Solving the optimal control problems is, however, time consuming. This means that the game optimal support times related to a certain initial state cannot be solved in real time as such. Hence, an approximate real time solution scheme based on linear interpolation of the off-line computed solutions of the optimal control problems is also introduced.

The paper is structured as follows. In the following section, essentials of a missile duel are described. The support time game formulation and the real time solution scheme are introduced in Sec. III. The game model is demonstrated with numerical examples in Sec. IV, followed by discussion and conclusions in Secs. V and VI, respectively.

II. Missile Duel

As stated in the Introduction, we consider a missile duel between two aircraft that are already committed to a combat without the possibility to disengage from the situation. We assume that in the combat, each aircraft uses one missile having the support, extrapolation, and lock-on phases in its flight as described earlier. Once the aircraft are within each others' firing envelopes, both aircraft are assumed to fire the missile toward the target simultaneously.

In the support phase, the missile first receives target information via an uplink (see, e.g., [14]) from the launching aircraft that tracks the target by its own radar. The flight direction during the tracking is constrained by the maximal look angle of the aircraft's radar, known as the gimbal limit. After a certain support time, the aircraft ceases the target tracking, breaks the uplink, and starts to evade the missile possibly fired by the adversary. The end of the support phase usually occurs before the missile's seeker can lock on the target. Therefore, the missile extrapolates the target trajectory using the latest available target information. The phase continues until the missile reaches a certain lock-on distance to the extrapolated position of the target, whereupon it switches on its own radar with an attempt to lock on the target. The lock-on distance reflects the maximum range of the missile's radar where it establishes a track with a given probability [15]. The maximum tracking range depends, among others, on the radar cross section of the target that in turn depends on its alignment and other factors.

If the target is equipped with a radar warning receiver, the missile's lock-on to the target causes an alarm which is typically the first direct evidence of an incoming threat. The target then initiates evasive maneuvers. The target pilot may try to outrun the missile by making a hard break turn and diving (see [13], for computational analysis, see [3]). The break turn is usually accompanied by employment of electronic countermeasures such as noise jamming, chaff, flares, and decoys. By directing the break turn vertically nose down, the line-of-sight rate and look-down of the missile can be increased. This may force the missile's seeker to lose its track due to the seeker's tracking rate limitations and ground clutter. For miss-distance maximizing endgame maneuvers, see [16].

III. Support Time Game

In this section, we formulate the support time game between two pilots hereafter referred to as the blue and the red players. The game provides game optimal support times of the missiles for given initial states of the aircraft and the missiles.

A. Description of the Duel

At this stage, we make the following assumptions:

1) The players fire the missiles simultaneously.

2) The lock-on range of the missile seeker is constant.

3) The aircraft detects the closing missile only when it locks on.

4) During the extrapolation, the target position is extrapolated linearly.

5) No countermeasures are available.

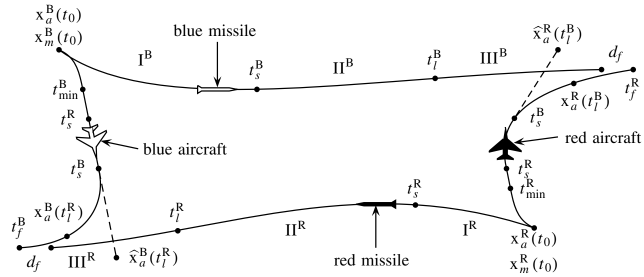

Fig. 1 One-on-one air combat with missiles. Dashed lines refer to the extrapolated trajectories of the targets.

6) The measured states of the vehicles are assumed accurate.

7) The players are so far away from each other and the support lasts such a time that irrespective of the target maneuvering, a constant flight direction keeps the target within the gimbal limit.

We next introduce the phases of the duel from the blue player's point of view under the assumptions above. The description from the red's point of view is obtained by switching the indices and the player names.

The phases of the duel are illustrated in Fig. 1. At t_0, the blue aircraft (BA) launches the blue missile (BM) toward the red aircraft (RA). Note that RA launches the red missile (RM) simultaneously, and BA does not detect RM yet. Immediately after the launch, BA starts to turn to the gimbal limit of its radar. At $t^B_{\min}$, BA reaches the gimbal limit and continues supporting BM until t^B_s. As stated above, the required flight direction at the limit is not affected by the adversary's maneuvering. The aircraft thus applies initial state dependent predetermined controls during the support phase. These controls are determined such that the aircraft makes a hard turn to the gimbal limit of its radar and stays at the limit until the end of the support phase t^B_s. Thus, blue's support phase I^B covers the interval $[t_0, t^B_s]$, where $t^B_s \geq t^B_{\min}$.

At t^B_s, BA finishes supporting BM, and consequently the missile stops receiving target information. BM then extrapolates the target trajectory with the latest available target information by assuming that the target flies at constant velocity and constant course. At t^B_l, BM's range to the extrapolated target position reaches the maximal lock-on distance d_l. There, BM switches on its radar and tries to lock on the target. The lock-on is successful with a probability modeled in Sec. III.D. Just before the lock-on, BM assumes that RA's position is $\hat{\mathbf{x}}^R_a(t^B_l)$. At t^B_l, BM updates the target position to RA's real position $\mathbf{x}^R_a(t^B_l)$. Hence, the extrapolation phase of blue II^B covers the interval $[t^B_s, t^B_l]$.

After locking on the target at t^B_l, BM begins to receive accurate target information again. RA evades the missile such that the probability of an effective hit is minimized. As described earlier, we assume that the probability depends on the closing velocity attained by BM (see again Sec. III.D). Therefore, the objective of RA is to minimize the closing velocity at the given final distance d_f that occurs at t^R_f. To summarize, the lock-on phase of blue III^B covers the interval $[t^B_l, t^R_f]$.

After the support phase, BA maneuvers according to a worst case scenario, in which RA launches RM at t_0, and the missile receives accurate target information for the total duration of its flight. At t^B_s, BA starts to evade this hypothetical missile launched at t_0. At t^R_l, the true RM locks on BA that detects the missile. Henceforth, BA evades the detected RM. At t^B_f, RM's range to BA is equal to the final distance d_f.

B. Dynamic Equations

1. Aircraft Model

The dynamics of the aircraft is described by a point-mass model, that is, by the following system of differential equations [17]:

$$\dot{x}_a = v_a \cos\gamma_a \cos\chi_a \tag{1}$$

$$\dot{y}_a = v_a \cos\gamma_a \sin\chi_a \tag{2}$$

$$\dot{h}_a = v_a \sin\gamma_a \tag{3}$$

$$\dot{\gamma}_a = \frac{1}{m_a v_a}\{[L_a((\alpha, h_a, v_a, M(h_a, v_a)) + \eta T_{\max}(h_a, M(h_a, v_a))\sin\alpha]\cos\mu - m_a g\cos\gamma_a\} \tag{4}$$

$$\dot{\chi}_a = \frac{1}{m_a v_a \cos\gamma_a}[L_a(\alpha, h_a, v_a, M(h_a, v_a)) + \eta T_{\max}(h_a, M(h_a, v_a))\sin\alpha]\sin\mu \tag{5}$$

$$\dot{v}_a = \frac{1}{m_a}[\eta T_{\max}(h_a, M(h_a, v_a))\cos\alpha - D_a(\alpha, h_a, v_a, M(h_a, v_a))] - g\sin\gamma_a \tag{6}$$

where x_a and y_a refer to the horizontal coordinates and h_a to the altitude of the aircraft. The remaining three state variables are the flight path angle γ_a, the heading angle χ_a, and the velocity v_a. The heading angle is the angle between the x axis and the projection of the velocity vector on the xy plane, whereas the flight path angle is the angle between the projection and the velocity vector itself. The aircraft is guided with the angle of attack α, the throttle setting η, and the bank angle μ. The first two variables control the normal and tangential accelerations of the aircraft whereas the last one produces a horizontal turn by directing the aircraft's normal force away from a vertical plane. The aircraft is also subject to a set of control and state constraints given in the Appendix.

The acceleration due to the gravity as well as the mass of the aircraft, denoted by g and m_a, respectively, are assumed constant. $T_{\max}(\cdot)$ denotes the maximum available thrust force, whereas $M(\cdot)$ is the Mach number. The lift force $L_a(\cdot)$ and the drag force $D_a(\cdot)$ are detailed in the Appendix.

2. *Missile Model*

The motion of the missile is described by

$$\dot{x}_m = v_m \cos\gamma_m \cos\chi_m \tag{7}$$

$$\dot{y}_m = v_m \cos\gamma_m \sin\chi_m \tag{8}$$

$$\dot{h}_m = v_m \sin\gamma_m \tag{9}$$

$$\dot{\gamma}_m = \frac{1}{v_m}(a_1 - g\cos\gamma_m) \tag{10}$$

$$\dot{\chi}_m = \frac{1}{v_m \cos\gamma_m} a_2 \tag{11}$$

$$\dot{v}_m = \frac{1}{m_m(t)}[T_m(t) - D_m(a, h_m, v_m, M(h_m, v_m))] - g\sin\gamma_m \tag{12}$$

$$\dot{a}_1 = \frac{1}{\tau}(a_{1c} - a_1) \tag{13}$$

$$\dot{a}_2 = \frac{1}{\tau}(a_{2c} - a_2) \tag{14}$$

The interpretation of the first six state variables is similar to those of the aircraft model. The remaining two state variables a_1 and a_2 are the states of the autopilot, which is here modeled as a first-order system. The parameter τ is the time constant of the autopilot. The states denote the vertical and horizontal acceleration components of the missile that are orthogonal to the velocity vector of the missile. The commanded accelerations a_{1c} and a_{2c} depend on the guidance law. Commands resulting from proportional navigation are described in the Appendix and used in the numerical examples.

The mass of the missile $m_m(t)$ and the thrust force $T_m(t)$ are given as tabular data. The drag force $D_m(\cdot)$ is given in the Appendix.

The sets of differential equations (1–6) and (7–14) can be written in shorthand for player $i = B, R$ as

$$\dot{\mathbf{x}}^i_a = \mathbf{f}_a\left(\mathbf{x}^i_a, \mathbf{u}^i\right), \qquad \mathbf{x}^i_a(t_0) = \mathbf{x}^i_{a_0} \tag{15}$$

and

$$\dot{\mathbf{x}}^i_m = \mathbf{f}_m\left(\mathbf{x}^i_m, t\right), \qquad \mathbf{x}^i_m(t_0) = \mathbf{x}^i_{m_0} \tag{16}$$

where

$$\mathbf{x}^i_a = \left[x^i_a \ y^i_a \ h^i_a \ \gamma^i_a \ \chi^i_a \ v^i_a\right]^T$$

and

$$\mathbf{x}^i_m = \left[x^i_m \ y^i_m \ h^i_m \ \gamma^i_m \ \chi^i_m \ v^i_m \ a^i_1 \ a^i_2\right]^T$$

are the state vectors of the aircraft and the missile, $\mathbf{u}^i = [\alpha^i \ \eta^i \ \mu^i]^T$ is the control vector of the aircraft, and $\mathbf{x}^i_{a_0}$ as well as $\mathbf{x}^i_{m_0}$ are the initial states of the aircraft and the missile, respectively.

C. Payoff Functions

The pilot's conflicting goals, maximization of the probabilities of survival and hit, can be combined into a single payoff in a number of ways. A standard way to produce efficient or Pareto optimal solutions is to aggregate the single components into a weighted sum [18], which is used here, too. In this specific problem, the weights have an interpretation as a description of the risk attitude of the decision maker. Also multiplicative forms, or maximization of one probability under the constraint that the value of the other stays above a given level (see, e.g., [9]), can be used.

Hence, given the initial states of the players, blue maximizes the payoff

$$J^B\left(\mathbf{x}_0, t^B_s, t^R_s; w^B\right) = w^B p^B_h\left(\mathbf{x}_0, t^B_s, t^R_s\right) + (1 - w^B) \times \left[1 - p^R_h\left(\mathbf{x}_0, t^B_s, t^R_s\right)\right], \qquad w^B \in [0, 1] \tag{17}$$

where $\mathbf{x}_0 = [\mathbf{x}^{B^T}_{a_0} \ \mathbf{x}^{R^T}_{a_0}]^T$. Note that for a single player, the initial states of the aircraft and the missile coincide and the initial states of the missile's autopilot are zero. For clarity, we suppress the explicit dependence on the initial states in the following. Because of the previous assumptions, the controls during the support and optimal missile evasion depend on the initial states and the support times and do not appear explicitly. The first term contains the probability of an effective hit achieved by blue multiplied by the weight w^B. The latter term is the complement of red's probability of an effective hit, that is, the survival probability of blue. The larger the weight w^B, the more risk prone blue is.

In this paper, we presume that the probability of an effective hit is the joint probability of two events: the probability that the missile is able to lock on the target, and the probability that the missile reaches the target assumed it has locked on. The former is commonly known as the probability of guidance, whereas the latter is called here the probability of reach. The overall probability of BM's hit to RA is then given as

$$p^B_h\left(t^B_s, t^R_s\right) = p^B_g\left(t^B_s, t^R_s\right) p^B_r\left(t^B_s, t^R_s\right) \tag{18}$$

where $p^B_g(t^B_s, t^R_s)$ is BM's probability of guidance and $p^B_r(t^B_s, t^R_s)$ is the probability of reach for BM. The payoff of red is formulated similarly by switching the names and indices. Note that in this analysis, we do not consider probability of kill that commonly contains also the missile warhead operation.

D. Probabilities of Guidance and Reach

1. Probability of Guidance

In the lack of a detailed seeker model, we are forced to generalizing assumptions in the modeling of the probabilities of guidance and reach. Therefore, BM's probability of guidance is modeled to depend linearly on the tracking error accumulated during the extrapolation phase. The probability is defined as

$$p_g^B\left(t_s^B, t_s^R\right) = 1 - \min\left\{\frac{\theta^B\left(t_s^B, t_s^R\right)}{\theta_{\max}}, 1\right\} \tag{19}$$

where the tracking error $\theta^B(t_s^B, t_s^R)$ is the angle between the true line-of-sight vector LOS and the estimated vector $\widehat{\text{LOS}}$ from BM to the real $\mathbf{x}_a^R$ and estimated position $\hat{\mathbf{x}}_a^R$ of RA at t_l^B (see Fig. 2). The lock-on time of BM, denoted by

$$t_l^B = \min\left\{t\colon d\left(\mathbf{x}_m^B(t), \hat{\mathbf{x}}_m^R(t)\right) \le d_l, \qquad t \in \left[t_0, t_f^R\right]\right\} \tag{20}$$

is the earliest moment when the range to the estimated target position reaches the given lock-on distance d_l, and $\theta_{\max}$ is the limit angle after which the probability of guidance becomes zero. The tracking error is given by

$$\theta^B\left(t_s^B, t_s^R\right) = \arccos\left\{\left[\left(x_a^R - x_m^R\right)\left(\hat{x}_a^B - x_m^B\right) + \left(y_a^R - y_m^R\right) \times \left(\hat{y}_a^B - y_m^B\right) + \left(h_a^R - h_m^R\right)\left(\hat{h}_a^B - h_m^B\right)\right]\Big/\left(dd_l\right)\right\} \tag{21}$$

where the true distance between the missile and the target equals

$$d = \sqrt{\left(x_a^R - x_m^B\right)^2 + \left(y_a^R - y_m^B\right)^2 + \left(h_a^R - h_m^B\right)^2} \tag{22}$$

The estimated position of RA is obtained by integrating the state equations (1–3) with $\dot{v}_a^R = \dot{\gamma}_a^R = \dot{\chi}_a^R = 0$ for $t \in [t_s^B, t_l^B]$. That is, the target is assumed to fly with constant velocity and course whereas the missile is guided by its guidance law, or a specific midcourse guidance law. Note that for a nonmaneuvering target, proportional navigation leads to a direct course toward the expected impact point [19].

In reality, the probability of guidance depends, among others, on the launch range, quality of the uplink between the supporting aircraft and the missile, radar cross section of the target, closing velocity and direction, tracking error, type of the seeker head, and weather conditions. Nevertheless, a seeker model would be needed to model these effects. If a seeker model is available, it can be utilized as a part of the game model.

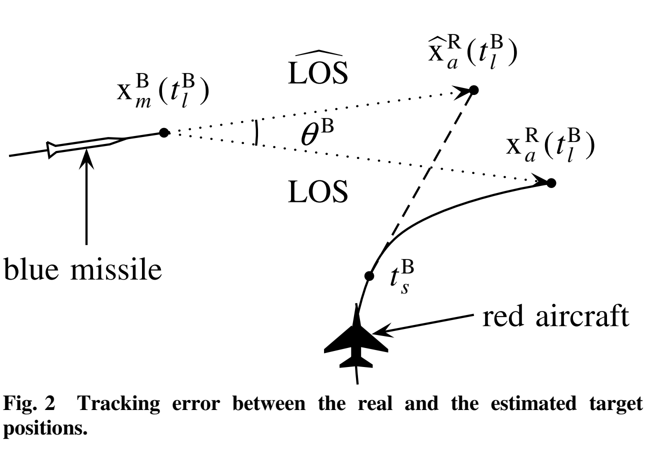

Fig. 2 Tracking error between the real and the estimated target positions.

2. Probability of Reach

The studies of endgame analysis (see, e.g., [16]) show that the ability of the target to avoid the missile depends strongly on the closing velocity of the missile. Here, we model the BM's probability of reach simply as

$$p_r^B\left(t_s^B, t_s^R\right) = \min\left\{\frac{v_c^R\left(t_f^R\right)}{v_{c,\max}}, 1\right\} \tag{23}$$

where $v_c^R(t_f^R)$ is the closing velocity of BM to RA when the distance between the missile and the aircraft is d_f. At closing velocities $v_{c,\max}$ or greater, the probability of reach equals one.

3. Minimum Closing Velocity

Having modeled the probability of reach as above, it is evident that a player wishes to minimize the closing velocity of the missile. For each support time pair of the players, the minimal closing velocity of a player-missile pair at the final distance d_f is obtained as the solution to the following free final time optimal control problem, given here for red:

$$\min_{\mathbf{u}^R, t_f^R} v_c\left(t_f^R\right) \tag{24}$$

$$\text{s.t. } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}^R, t), \qquad \mathbf{x}\left(t_s^R\right) = \left[\mathbf{x}_a^{R^T}\left(t_s^R\right) \quad \mathbf{x}_m^{B^T}\left(t_s^R\right)\right]^T \quad t\left[t_s^R, t_f^R\right] \tag{25}$$

$$\mathbf{g}\left(\mathbf{x}, \mathbf{u}^R\right) \le \mathbf{0} \tag{26}$$

$$d\left(t_f^R\right) - d_f = 0 \tag{27}$$

where the state vector

$$\mathbf{x} = \left[\mathbf{x}_a^{R^T} \quad \mathbf{x}_m^{B^T}\right]^T$$

consists of the states of RA and BM, respectively. The initial state of the problem $\mathbf{x}(t_s^R)$ consists of the states of RA and BM at the end of the red player's support phase. The differential equations (25) describing the dynamics of the vehicles refer to (15) with $i = R$ and (16) with $i = B$. Constraints (26) limiting the controls and preventing, for example, stalling and exposure to excessive accelerations refer to (A4–A9) and (A18) given in the Appendix. Equation (27) fixes the final distance between the aircraft and the missile.

E. Solution of the Support Time Game

Because the players are noncooperative, rational, and the players are assumed to know each others' payoff functions given by Eq. (17), the players end up playing a Nash equilibrium. It can be obtained, for example, by the following best response algorithm with an appropriate starting point t_0 and the desired accuracy ε:

1) Set $t_0^B := t_0$, $t_0^R := t_0$, and $k := 0$.
2) Determine the maximizing support time of blue: $t_{k+1}^B := \arg\max_{t^B} J^B(\mathbf{x}_0, t^B, t_k^R; w^B)$.
3) Determine the maximizing support time of red: $t_{k+1}^R := \arg\max_{t^R} J^R(\mathbf{x}_0, t_{k+1}^B, t^R; w^R)$.
4) Set $k := k + 1$.
5) If $|t_k^B - t_{k-1}^B| < \varepsilon$ and $|t_k^R - t_{k-1}^R| < \varepsilon$, stop. Otherwise, go to step 2.

It is possible that not a single or more than one pure strategy Nash equilibria exist. Sufficient conditions for the existence of an equilibrium are that the strategy sets are compact and convex, and the payoff functions to be maximized are continuous and strictly concave (see [1]). Here, the strategy sets, being closed time intervals, are clearly compact and convex. The payoff functions are continuous, but their concavity is hard to establish. The concavity depends on the weights $w^i, i = B, R$, too. Nevertheless, if the aforementioned algorithm terminates, the convergence point is a Nash equilibrium solution of the support time game. In case of multiple equilibria, the algorithm converges into one of them depending on the starting point of the iteration. Hence, the possibility of multiple solutions can be examined by starting the algorithm with different initial values t_0 and checking whether the iterations converge into different time points.

1. Evaluation of the Payoffs

The evaluation of the blue and the red player's payoffs in the best response algorithm would require a solution of the closing-velocity minimizing optimal control problem numerous times. Here, an alternative approach is taken. For given initial states of the aircraft and missiles, the probabilities of guidance and reach are determined for a set of support time pairs $(t_s^B, t_s^R) \in T^B \times T^R$, where $T^i = \{t^i_{\min}, t^i_2, \ldots, t^i_n\}$, $i = B, R$ in advance. Then, the probabilities are approximated with suitable continuous and smooth functions. Thus, for given weights w^B and w^R as well as support times t_s^B and t_s^R, the payoffs can be evaluated simply by using these approximations in Eq. (17). The method has an additional benefit that reevaluation of the payoffs with different weights and parameters of the probabilities of guidance and reach does not require resolving the optimal control problems.

The optimal control problems for the minimal closing velocities are solved numerically with a direct multiple shooting method. The problem is transformed into a nonlinear optimization problem by discretizing it with respect to time and parameterizing the controls of the aircraft. The resulting problem is solved with sequential quadratic programming [20]. The solution method is described thoroughly in [16].

F. Real Time Solution Scheme

The obtained game optimal support times of the missiles apply only for a single initial state of the duel. Because solving the optimal control problems is time consuming, the game cannot be solved in real time as such. However, game optimal support times related to a current state of the combat should be readily available in reality. Motivated by this, we next introduce a scheme for obtaining an approximate real time solution of the game as a function of the initial state of the duel.

As such, the initial state space consists of the states of both aircraft adding up to 12 state variables. However, because the atmosphere is assumed laterally homogenous, only the lateral distance is relevant and we can fix

$$x_0^B = y_0^B = y_0^R = 0 \tag{28}$$

Because the initial range is relatively long, the initial values of the flight path angles are considered negligible and we also use

$$\gamma_0^B = \gamma_0^R = 0 \tag{29}$$

The remaining seven-dimensional initial state space consists of the velocities v_0^i, the heading angles χ_0^i, and the altitudes h_0^i of the aircraft $i = B, R$, as well as the lateral distance x_0^R.

At first, the minimum closing velocities and the tracking errors corresponding to the sets of support times T^i of the players $i = B, R$ are solved off-line for a set of initial states $S = S_1 \times \cdots \times S_7$, where $S_1 = \{v^B_{0,k}\}^{n_v}_{k=1}$, $S_2 = \{v^R_{0,k}\}^{n_v}_{k=1}$, $S_3 = \{\chi^B_{0,k}\}^{n_\chi}_{k=1}$, $S_4 = \{\chi^R_{0,k}\}^{n_\chi}_{k=1}$, $S_5 = \{h^B_{0,k}\}^{n_h}_{k=1}$, $S_6 = \{h^R_{0,k}\}^{n_h}_{k=1}$, and $S_7 = \{x^R_{0,k}\}^{n_r}_{k=1}$ are ordered sets.

Second, the support times and the corresponding minimum closing velocities and tracking errors are linearly interpolated for a given intermediate initial state between the nodes of S.

Finally, approximate game optimal support times are obtained by evaluating the payoffs on the basis of the interpolated support times, closing velocities, and tracking errors, and by applying the best response algorithm. Although the off-line computation of the closing velocities is time consuming, the interpolation and the best response iteration can be performed in real time.

IV. Numerical Examples

In this section, the support time game is demonstrated with two numerical examples. In the first example, the game is solved for a single initial state for a set of cases where red's risk attitude is kept constant and blue's risk attitude is varied from risk averse to risk prone. In the second example, the game is solved for several initial states by utilizing the real time solution scheme.

The initial altitudes are chosen relatively high to be able to perform the support and evasive maneuvers in a vertical plane. In practice, this is preferred because due to the gravity, the turn rate and the velocity of the aircraft are increased most efficiently by diving, and the aircraft can achieve larger thrust in lower altitude (see [13]). Diving also increases look-down of the missile that impedes the operation of the missile's seeker as mentioned in Sec. II. Altogether, diving improves the chance of a successful outrun without affecting the target tracking capacity. Note that the game model permits arbitrary selection of the controls for the support phase. Thus, in case of low altitudes, the support maneuver could be performed, for example, laterally.

The aircraft and the missile parameters are the same for both players. The maximum angle of attack, the minimum altitude, the maximum dynamic pressure, and the maximum load factor of the aircraft are set to $\alpha_{\max} = 32$ deg, $h_{a,\min} = 1000$ m, $q_{\max} = 80$ kPa, and $n_{a,\max} = 9$, respectively. The minimum altitude and the maximum load factor of the missiles are set to $h_{m,\min} = 1000$ m and $n_{m,\max} = 40$, respectively. The missile has a boost-sustain propulsion system, and the thrust history of the missile is of the form

$$T_m(t) = \begin{cases} T_b, & 0 \le t \le 3 \text{ s} \\ T_s, & 3 \le t \le 8 \text{ s} \\ 0, & t > 8 \text{ s} \end{cases} \tag{30}$$

Consequently, the mass of the missile $m_m(t)$ first decreases piecewise linearly and remains thereafter constant. The aircraft employs an afterburner.

A. Example 1

At launch time, the states of the aircraft and the missile are the same for a single player. The initial states are shown in Table 1.

The initial distance between the players is chosen such that the closing velocities of the missiles to their targets are positive even if the targets start to evade as soon as possible. A hard turn to the gimbal limit is performed as follows. First, the aircraft is inverted by changing the bank angle to 180 deg and increasing the angle of attack to its maximum. After the desired flight path angle is achieved, the angle of attack is chosen such that the course of the aircraft remains unchanged till the end of the support phase t_s^i, $i = B, R$.

We assume that the gimbal limits of the radars of BA and RA are 60 deg. When the players perform the aforementioned turning maneuver for 5.0 s, the flight path angles of BA and RA reach $\gamma^B = -59.5$ deg and $\gamma^R = -59.1$ deg for the rest of the support maneuver, that is, the players fly near their gimbal limits. The closing velocities and tracking errors are computed for every pair of support times in $T^B \times T^R$, where $T^B = T^R = \{5.0, 7.0, 9.0, 11.0, 13.0, 15.0\}$. The final distance and the lock-on distance of the missiles

Table 1 Initial states of the aircraft and the missiles

i	x_0^i, m	y_0^i, m	h_0^i, m	γ_0^i, deg	χ_0^i, deg	v_0^i, m/s	$a^i_{1_0}$, m/s^2	$a^i_{2_0}$, m/s^2
B	0.0	2000.0	9750.0	0.0	0.0	275.0	0.0	0.0
R	18,000.0	2000.0	9500.0	0.0	180.0	250.0	0.0	0.0

Table 2 Game optimal support times for the different weights of blue with $w^R = 0.5$. Number of iterations with tolerance $\varepsilon = 0.01$ is given by No. of its

w^B	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$t_s^{B^*}$, s	5.00	5.00	5.00	6.39	8.67	10.03	10.98	11.74	12.38	12.94	13.46
$t_s^{R^*}$, s	11.75	11.75	11.75	10.84	10.16	10.09	10.14	10.21	10.29	10.38	10.46
No. of its	3	3	3	5	4	4	4	4	4	4	4

are set to $d_f = 500$ m and $d_l = 4000$ m, respectively. The threshold values for the closing velocity and the tracking error are set to $v_{c,\max} = 600$ m/s and $\theta_{\max} = 10$ deg, respectively. The values of the parameters are only suggestive.

The weight of the RA's payoff function is set to $w^R = 0.5$, that is, red considers the probabilities of survival and hit equally important. The weights of blue are chosen from $w^B \in \{0, 0.1, \ldots, 1.0\}$. Game optimal support times of the players are solved for each weight of blue. The solutions, as well as the number of iteration steps needed to converge to a Nash equilibrium with tolerance $\varepsilon = 0.01$, are presented in Table 2. The reaction curves of the players are shown in Fig. 3. The reaction curve gives the player's optimal support times for all the support times of the adversary.

The reaction curves of blue shift from left to right as the weight w^B increases from zero to one. Large weights correspond to preferring hit to the adversary to own survival, and so the missile is supported for a longer time in these cases. With small weights, own survival is preferred, which means that the player initiates the evasion earlier. Note that when $w^B \in \{0, 0.1, 0.2\}$, blue always chooses $t_s^{B^*} = 5.0$ s.

When the support time of red is small, optimal support times of blue tend to be large regardless of how much blue prefers the probability of survival. The reason for this is that although red supports the missile only for a short time, RM's probability of guidance is small irrespective of the support time of blue. In that case, RM's probability of hit is small, and BM's probability of hit to RA contributes strongly to blue's payoff. Thus, it is worthwhile for blue to increase the probability of hit to red by supporting the missile for a longer time.

Figure 4 presents a three-dimensional graph of blue's payoff with $w^B = 0.5$ and $w^R = 0.5$. The figure reveals that deviation from the reaction curve, represented by the dashed curve in Fig. 3, reduces the payoff of blue. With $t_s^R = 15.00$ s, the difference between the best and the worst payoff of blue is maximal. Then, in the reaction curve, the probabilities of hit of BM and RM are $p_h^B = 0.92$ and $p_h^R = 0.71$, whereas the payoff of blue is $J^B = 0.61$. In the worst case, that is, when $t^B = 5.00$ s, the above probabilities equal $p_h^B = 0.31$ and $p_h^R = 0.42$, whereas $J^B = 0.45$. The difference between the optimal and the worst payoff is 0.16 units which is over 26% of the optimal payoff. Hence, the optimization of the payoff is worth the effort.

Figure 5 shows the optimal trajectories of the vehicles for $w^B = 0$ and $w^R = 0.5$. In the figure, the dashed lines refer to the missile trajectories in the extrapolation phases, and the shade of the ribbon denotes the velocity of the aircraft, the darker shade referring to a higher velocity. Now, blue is extremely risk averse, that is, he considers only his own survival. Game optimal support times of the players are $t_s^{B^*} = 5.0$ s and $t_s^{R^*} = 11.75$ s, whereas the extrapolation times of BM and RM are 8.70 and 3.59 s, respectively. The long extrapolation time of BM decreases the missile's probability of guidance to $p_g^B = 0.31$, whereas RM's shorter extrapolation time results in the probability of guidance of $p_g^R = 0.96$. On the other hand, the early evasion of BA decreases the probability of reach of RM to $p_r^R = 0.45$, whereas the longer support maneuver of RA increases the probability of reach of BM to $p_r^B = 0.87$. The

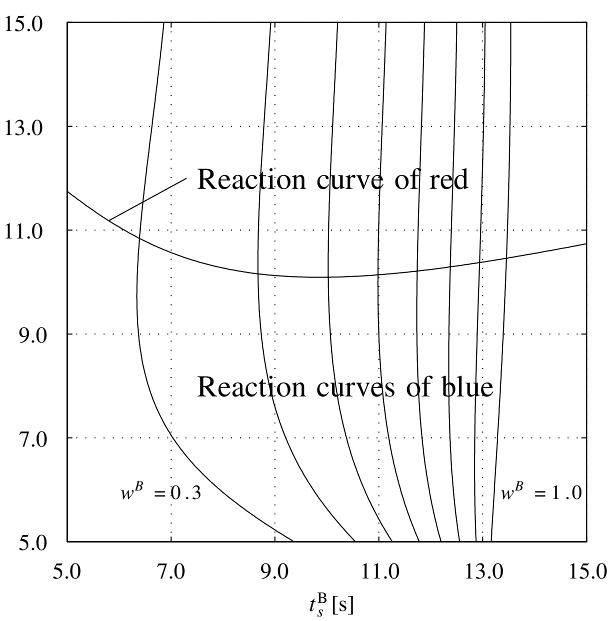

Fig. 3 The reaction curves of the players for $w^B = \{0, 0.1, \ldots, 1.0\}$ and $w^R = 0.5$.

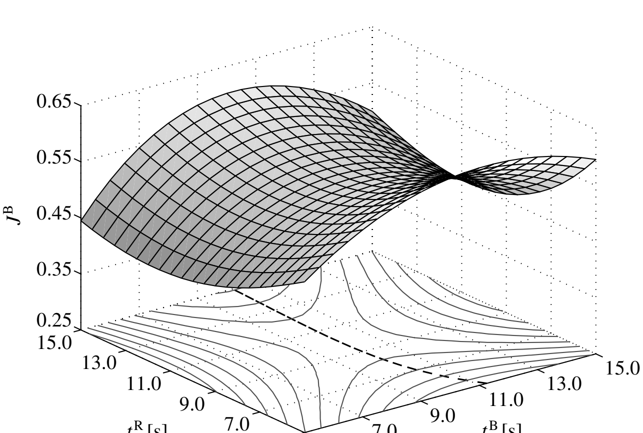

Fig. 4 Three-dimensional graph of blue's payoff with $w^B = 0.5$ and $w^R = 0.5$.

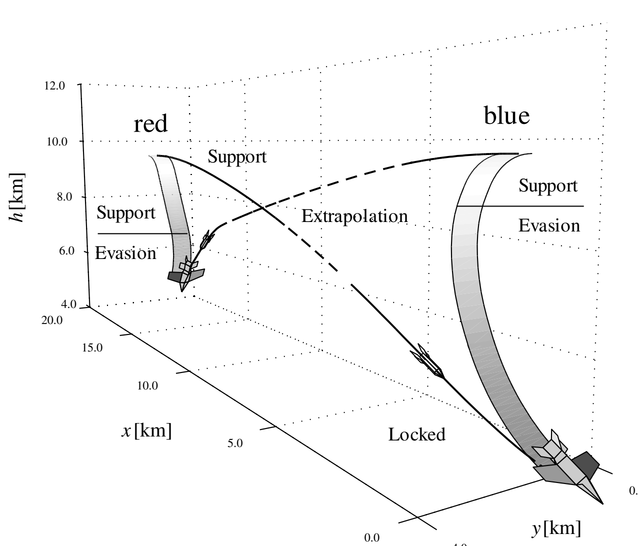

Fig. 5 The optimal trajectories of the vehicles when $w^B = 0.5$ and $w^R = 0.5$.

Table 3 The values of the initial state variables in S

i	v_0^i, m/s	χ_0^i, deg	h_0^i, m	x_0^R, m
B	{275.0, 300.0}	{0.0, 30.0}	{9500.0, 10, 000.0}	{17, 000.0, 17, 500.0, 18, 000.0}
R	{250.0, 275.0}	{150.0, 180.0}	{9500.0, 10, 000.0}	——

Table 4 Initial states for the real time solution scheme

i	v_0^i, m/s	χ_0^i, deg	h_0^i, m	x_0^R, m
B	275.0	0.0	9750.0	{17, 000.0, 17, 250.0, 17, 500.0, 17, 750.0, 18, 000.0}
R	250.0	180.0	9500.0	——

probabilities of hit of BA and RA are $p_h^B = 0.27$ and $p_h^B = 0.43$, respectively, whereas the payoffs are $J^B = 0.57$ and $J^R = 0.58$.

B. Example 2

The sets of initial velocities, heading angles, altitudes, and lateral distances applied in the construction of the set of initial states S are shown in Table 3. Here, S consists of 192 different initial states. For illustrative purposes, S is here relatively small.

After solving the optimal control problems related to each initial state of S off-line, the real time solution scheme is applied for the set of initial states presented in Table 4. Note that the initial states are not elements of S, and hence interpolation with respect to the altitude of BA and the lateral distance is required.

The reaction curves for the different initial states as well as the corresponding game optimal support times are shown in Fig. 6 and Table 5, respectively. The results indicate that the longer the initial distance between the players, the longer the players support their missiles. This is reasonable because the closing velocity of the missile at a fixed final distance obviously decreases when the launch distance is increased. This enables longer support time without increasing the probability of reach of the adversary's missile.

Note that with the largest value of x_0^R, the initial state corresponds to that of the first example. Comparisons between the corresponding reaction curves and game solutions in examples 1 and 2 reveal that the optimal reaction times and the game optimal support times of the players are slightly larger in the first example, especially for the red player. This is due to the errors caused by linear interpolation.

To analyze the magnitude of the approximation error, the game is solved with a new initial state defined as $v_0^B = 287.5$ m/s, $v_0^R = 262.5$ m/s, $\chi_0^B = 15$ deg, $\chi_0^R = 165$ deg, $h_0^B = 9750$ m, $h_0^R = 9750$ m, and $x_0^R = 17,750$ m. Note that the initial state is chosen such that interpolation must be performed with respect to each of the seven state variables in the real time solution scheme. Consequently, the approximation error is presumably large. Game optimal support times are $t_s^{B^*} = 9.13$ s and $t_s^{R^*} = 10.03$ s, whereas the approximate support times obtained with the real time solution scheme are $t^{B^\circ} = 9.74$ s and $t^{R^\circ} = 9.27$ s. The absolute differences between the corresponding support times are 0.61 and 0.75 s for blue and red, respectively. Obviously, the above analysis does not provide an absolute bound for the approximation error, but it gives an insight about the magnitude of the approximation error at worst. The curvatures of the players' payoffs for arbitrary initial states are typically similar to that of Fig. 4. The figure indicates that the payoffs are not too sensitive to perturbations from the reaction curve. Note that the magnitude of the approximation error could be reduced by using a denser set of initial states in the real time solution scheme.

Table 5 Approximate game optimal support times for a set of lateral distances x_0^R with $w^B = 0.5$ and $w^R = 0.5$. Number of iterations with tolerance $\varepsilon = 0.01$ is given by No. of its

x_0^R	17,000.0	17,250.0	17,500.0	17,750.0	18,000.0
$t_s^{B^\circ}$, s	8.82	8.97	9.11	9.54	10.01
$t_s^{R^\circ}$, s	8.62	8.78	8.93	9.13	9.59
No. of its	4	4	4	4	4

V. Discussion

The underlying assumption of the support time game is that both players behave rationally and optimize their support times. In reality, this may not be the case due to, for example, psychological factors: a pilot may be eager to evade before the optimal moment under the threat of the enemy missile [13]. Presuming that the adversary does not employ his game optimal support time and the other pilot can observe this, he can use the game model by simply choosing the optimal support time from his reaction curve.

Being the first attempt to determine the game optimal support time of a medium-range air-to-air missile, the model could also be extended in a number of ways. For instance, the modeling of the probability of guidance could be improved by using a realistic seeker model. On the other hand, the support time game could be extended to cover also variable launch times. In that case, the launch should be performed during the break turn and the launch time would be chosen from a predefined set. This would result in a bimatrix game, whose payoffs would be determined by solving the support time game with the chosen launch times of the players. Game optimal launch times could be obtained as a Nash equilibrium solution of this bimatrix game.

Solving the optimal control problems needed in the determination of the probabilities of guidance and reach is time consuming, which means that game optimal support times related to a given initial state cannot be solved in real time as such. Here, the dilemma is

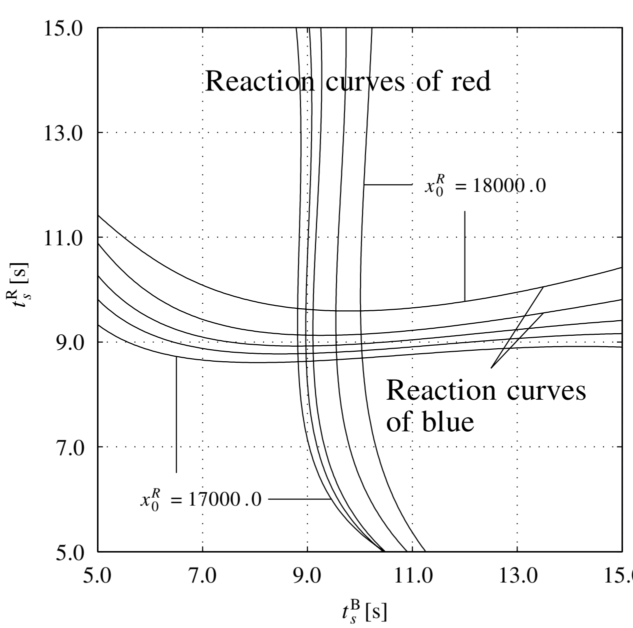

Fig. 6 The reaction curves of the players for the weights $w^B = w^R = 0.5$ and the lateral distances $x_0^R = \{17, 000.0, 17, 250.0, 17, 500.0, 17, 750.0, 18, 000.0\}$.

circumvented by solving the optimal control problems off-line for a set of initial states and interpolating the solutions for a given intermediate initial state. Thereafter, the game is solved by applying best response iteration on the basis of the interpolated minimum closing velocities and tracking errors. In example 2, the set of initial states for which the minimum closing velocities and the tracking errors are solved off-line is fairly modest. However, by applying parallel computation in the off-line phase, larger sets of initial states can be used.

VI. Conclusion

The game model presented in this paper is the first one in the open literature where the support times of the players are considered explicitly. Suitable assumptions on the players' behavior during the support phase decouple the dynamics of the game, allowing transformation of an intractable differential game into a tractable static game. The main contribution of this paper is the introduction of a tractable model for obtaining optimal support time of a missile. The paper also presents a real time solution scheme that provides approximate game optimal support times of the missiles as a function of the initial state of the duel. The scheme could be used, for example, in an unmanned aerial vehicle, in the guidance model of an air combat simulator, or in a pilot advisory system.

Appendix

The aircraft and missile models correspond to a generic fighter aircraft and a medium range air-to-air missile. The coefficients of the lift and drag forces as well as the thrust forces are given as tabular data. They are approximated with suitable continuously differentiable functions.

Aircraft Model

The lift force is given by

$$L_a(\alpha, h_a, v_a, M(h_a, v_a)) = C_{L_a}(\alpha, M(h_a, v_a))S_a q(h_a, v_a) \quad \text{(A1)}$$

where $C_{L_a}(\cdot)$ is the lift coefficient and S_a the reference wing area of the aircraft. The dynamic pressure is

$$q(h_a, v_a) = \tfrac{1}{2}\varrho(h_a)v_a^2 \quad \text{(A2)}$$

where the air density $\varrho(h_a)$ is taken from the International Standard Atmosphere. The drag force is of the form

$$D_a(\alpha, h_a, v_a, M(h_a, v_a)) = C_{D_a}(\alpha, M(h_a, v_a))S_a q(h_a, v_a) \quad \text{(A3)}$$

where $C_{D_a}(\cdot)$ denotes the total drag coefficient of the aircraft.

The control variables are constrained as

$$0 \le \alpha \le \alpha_{\max}, \qquad 0 \le \eta \le 1, \qquad -180 \le \mu \le 180 \text{ deg} \quad \text{(A4)}$$

To avoid stall, the angle of attack must be chosen so that the lift coefficient does not exceed aircraft specific value $C_{L_a,\max}(\cdot)$ at a given altitude and velocity, that is,

$$C_{L_a}(\alpha, M(h_a, v_a)) - C_{L_a,\max}(M(h_a, v_a)) \le 0 \quad \text{(A5)}$$

The load factor expressed in the wind coordinate system as

$$n_a(\alpha, h_a, v_a) = \frac{L_a(\alpha, h_a, v_a, M(h_a, v_a))}{m_a g} \quad \text{(A6)}$$

is limited by the structure of the aircraft. This imposes another constraint related to the angle of attack, altitude, and velocity:

$$n_a(\alpha, h_a, v_a) - n_{a,\max} \le 0 \quad \text{(A7)}$$

In addition, the altitude and the dynamic pressure are constrained by

$$h_{a,\min} - h_a \le 0 \quad \text{(A8)}$$

and

$$q(h_a, v_a) - q_{\max} \le 0 \quad \text{(A9)}$$

where $h_{a,\min}$ and $q_{\max}$ refer to the minimum altitude and the maximum dynamic pressure of the aircraft, respectively.

Missile Model

The commanded accelerations are given as

$$a_{ic} = \begin{cases} a_{i\mathrm{PN}} \min\{a_{n,\max}, a_{C_{L,\max}}\}/a_{\mathrm{PN}}, & \text{if } a_{\mathrm{PN}} > \min\{a_{n,\max}, a_{C_{L,\max}}\} \\ a_{i\mathrm{PN}}, & \text{otherwise} \end{cases} \quad \text{(A10)}$$

where

$$a_{\mathrm{PN}} = \sqrt{a_{1\mathrm{PN}}^2 + a_{2\mathrm{PN}}^2} \quad \text{(A11)}$$

and the acceleration components $a_{i\mathrm{PN}}$, $i = 1, 2$ are given by a proportional navigation guidance scheme that tries to steer the missile so that the angular rate of the line-of-sight vector from the missile to the target is driven toward zero. That is,

$$a_{1\mathrm{PN}} = N_e v_c \dot{\lambda} \cdot \omega_1 + g \cos\gamma_m \quad \text{(A12)}$$

$$a_{2\mathrm{PN}} = N_e v_c \dot{\lambda} \cdot \omega_2 \quad \text{(A13)}$$

where N_e is the navigation constant, v_c is the closing velocity between the missile and the target, $\dot{\lambda}$ is the angular rate of the line-of-sight vector, and $\dot{\lambda} \cdot \omega_1$ as well as $\dot{\lambda} \cdot \omega_2$ give its projections to the directions of a_1 and a_2, respectively. The latter term of (A12) compensates the gravity. The commanded accelerations are limited to values not imposing structural damage or stall, that is, the total commanded acceleration is not allowed to exceed either of the following limits:

$$a_{n,\max} = g n_{m,\max} \quad \text{(A14)}$$

where $n_{\max}$ is the maximal load factor permitted by the structure of the missile and

$$a_{C_{L_m},\max} = C_{L_m,\max} S_m q(h_m, v_m)/m_m(t) \quad \text{(A15)}$$

where the stall limit $C_{L_m,\max}$ is assumed constant and S_m is the reference wing area of the missile.

The drag force of the missile is given by

$$D_m(a, h_m, v_m, M(h_m, v_m)) = C_{D_m}(a, M(h_m, v_m))S_m q(h_m, v_m) \quad \text{(A16)}$$

Note that because we have ignored the total angle of attack, the drag coefficient $C_{D_m}(\cdot)$ is a function of the total lateral acceleration

$$a = \sqrt{a_1^2 + a_2^2} \quad \text{(A17)}$$

and the structural constraint (A14) is expressed in the wind coordinate system.

The altitude of the missile is constrained by

$$h_{m,\min} - h_m \le 0 \quad \text{(A18)}$$

where $h_{m,\min}$ refers to the minimum altitude of the missile.

References

[1] Basar, T., and Olsder, G. J., *Dynamic Noncooperative Game Theory*, 2nd ed., Academic Press, London, 1995, pp. 179, 230–235.

[2] Law, A. M., and Kelton, W. D. (eds.), *Simulation Modeling and Analysis*, McGraw–Hill, New York, 1991, pp. 87–93.

[3] Raivio, T., "Capture Set Computation of an Optimally Guided Missile," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 6, 2001, pp. 1167–1175.

[4] Isaacs, R., *Differential Games*, Krieger, New York, 1975, pp. 8–13.
[5] Davidovitz, A., and Shinar, J., "Eccentric Two-Target Model for Qualitative Air Combat Game Analysis," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 3, 1985, pp. 325–331.
[6] Grimm, W., and Well, K. H., "Modelling Air Combat as Differential Game, Recent Approaches and Future Requirements," *Differential Games—Developments in Modeling and Computation*, edited by R. P. Hämäläinen and H. Ehtamo, Springer, Berlin, 1991, pp. 1–13.
[7] Järmark, B., "A Missile Duel Between Two Aircraft," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 4, 1985, pp. 508–513.
[8] Moritz, K., Polis, R., and Well, K. H., "Pursuit-Evasion in Medium-Range Air-Combat Scenarios," *Computers and Mathematics with Applications*, Vol. 13, Nos. 1–3, 1987, pp. 167–180.
[9] Shinar, J., Siegel, A. W., and Gold, Y. I., "On the Analysis of a Complex Differential Game Using Artificial Intelligence Techniques," *Proceedings of the 27th IEEE Conference on Decision and Control*, IEEE, Piscataway, NJ, 1988, pp. 1436–1441.
[10] Le Ménec, S., and Bernhard, P., "Decision Support System for Medium Range Aerial Duels Combining Elements of Pursuit-Evasion Game Solutions with AI Techniques," *Annals of the International Society of Dynamic Games*, edited by G. J. Olsder, Vol. 3, New Trends in Dynamic Games and Applications, Birkhäuser, Boston, 1995, pp. 207–226.
[11] Neuman, F., "On the Approximate Solution of Complex Combat Games," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 1, 1990, pp. 128–136.
[12] Lazarus, E., "The Application of Value-Driven Decision-Making in Air Combat Simulation," *Proceedings of the IEEE International Conference on Systems, Man, and Cybernetics*, IEEE, Piscataway, NJ, 1997, pp. 2302–2307.
[13] Shaw, R. L., *Fighter Combat: Tactics and Maneuvering*, Naval Institute Press, Annapolis, MD, 1985, pp. 59, 395–396.
[14] Virtanen, K., Hämäläinen, R. P., and Mattila, V., "Team Optimal Signaling Strategies in Air Combat," *IEEE Transactions on Systems, Man, and Cybernetics—Part A: Systems and Humans*, Vol. 32, No. 4, July 2006, pp. 643–660.
[15] Skolnik, M. I., *Introduction to Radar Systems*, 3rd ed., McGraw–Hill, New York, 2001, pp. 88–94.
[16] Raivio, T., and Ranta, J., "Miss Distance Maximization in the Endgame," *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA, Reston, VA, 2002, pp. 1–11; also AIAA Paper 2002-4947.
[17] Miele, A., *Flight Mechanics, Vol. 1: Theory of Flight Paths*, Addison–Wesley, Reading, MA, 1962, pp. 48–50.
[18] Collette, Y., and Siarry, P., *Multiobjective Optimization, Principles and Case Studies*, Springer, Berlin, 2004, p. 45.
[19] Zarchan, P., "Tactical and Strategic Missile Guidance," *Progress in Astronautics and Aeronautics*, 3rd ed., AIAA, Reston, VA, 1997, Vol. 176, pp. 12–13.
[20] Bertsekas, D. P., *Nonlinear Programming*, Athena Scientific, Belmont, MA, 1995, pp. 372–382.